

Non-finitely based varieties of right alternative metabelian algebras

Alexey Kuz'min*

Abstract

Since 1976, it is known from the paper by V. P. Belkin that the variety RA_2 of right alternative metabelian (solvable of index 2) algebras over an arbitrary field is not Spechtian (contains non-finitely based subvarieties). In 2005, S. V. Pchelintsev proved that the variety generated by the Grassmann RA_2 -algebra of finite rank r over a field \mathcal{F} , for $\text{char}(\mathcal{F}) \neq 2$, is Spechtian iff $r = 1$. We construct a non-finitely based variety \mathfrak{M} generated by the Grassmann \mathcal{V} -algebra of rank 2 of certain finitely based subvariety $\mathcal{V} \subset \text{RA}_2$ over a field \mathcal{F} , for $\text{char}(\mathcal{F}) \neq 2, 3$, such that \mathfrak{M} can also be generated by the Grassmann envelope of a five-dimensional superalgebra with one-dimensional even part.

Key words: non-finitely based variety of algebras, Spechtian variety of algebras, right alternative metabelian algebra, superalgebra, Grassmann algebra.

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Introduction

A variety of algebras is said to be *Spechtian* (or to have the *Specht property*) if its every subvariety is finitely based. In 1986, A. R. Kemer [8, 9] solved the famous Specht problem [21] by proving that the variety of associative algebras over a field of characteristic 0 is Spechtian. A. Ya. Belov [2], A. V. Grishin [4], and V. V. Schigolev [18] constructed, independently, non-finitely based varieties of associative algebras over a field of prime characteristic.

The Specht property problems for varieties of nonassociative algebras are studied hard (see [1, 3, 5–7, 10–17, 22–26]). In 1968, M. R. Vaughan-Lee [25] proved the Specht property of the variety of metabelian Lie algebras over a field of characteristic distinct from 2. Also, in his work [26], M. R. Vaughan-Lee constructed a non-finitely based variety of Lie algebras over a field of characteristic 2. V. S. Drensky [3] generalized this result of [26] to the case of a field of arbitrary prime characteristic. Yu. A. Medvedev [11] proved the Specht property of the variety of metabelian Malcev algebras. U. U. Umirbaev [22] generalized this result of [11] to the case of metabelian binary-Lie algebras. Besides, in his work [23], U. U. Umirbaev proved the Specht property of every solvable variety of alternative algebras over a field of characteristic distinct from 2 and 3. The essentiality of

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these restrictions on the characteristic of a ground field is proved by Yu. A. Medvedev [12] and S. V. Pchelintsev [14].

There are analogs of the Kemer's Theorem [8] in the cases of Jordan, alternative, and Lie algebras over a field of characteristic 0. Namely, A. Ya. Vais and E. I. Zel'manov [24] proved that a finitely generated Jordan PI-algebra generates a Spechtian variety. A. V. Iltyakov obtained the similar results for alternative PI-algebras [5] and for finite dimensional Lie algebras [6]. Nevertheless, the Specht property problems for the varieties of all alternative, Lie, and Jordan algebras are still open.

Let \mathcal{F} be a field of characteristic distinct from 2. Consider the identities

$$(x, y, y) = 0 \quad (\text{the right alternative identity}), \quad (1)$$

$$(xy)(zt) = 0 \quad (\text{the metabelian identity}), \quad (2)$$

$$(x \circ y) \circ z = 0 \quad (\text{the identity of Jordan nilpotency of step 2}), \quad (3)$$

where $(a, b, c) = (ab)c - a(bc)$ is the associator of the elements a, b, c and $a \circ b = ab + ba$ is the Jordan product of the elements a, b . The variety RA_2 of right alternative metabelian algebras over \mathcal{F} is defined by (1), (2). By RA'_2 we denote the subvariety of RA_2 distinguished by (3).

Since 1976, it is known [1] that RA_2 is not Spechtian. I. M. Isaev [7] proved that non-finitely based subvarieties of RA_2 can even be generated by finite-dimensional algebras. Although it was not mentioned by the authors, the direct verification shows that the algebras constructed in [1, 7] satisfy (3), i. e. the referred results hold for RA'_2 as well. On the other hand, a number of corollaries of the Yu. A. Medvedev's Theorem on two-term identities [11] states the Specht property of the subvarieties of alternative, left-nilpotent, and $(-1, 1)$ -algebras in RA_2 . Certain generalizations of these results of [11] are obtained by the author in [10].

Recall [17] the notion of Grassmann \mathcal{V} -algebra of finite rank. Let \mathcal{V} be a variety of algebras over \mathcal{F} ; $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a *superalgebra* (\mathbb{Z}_2 -graded algebra) with the even part \mathcal{A}_0 and the odd part \mathcal{A}_1 , i. e. $\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j \pmod{2}}$ for $i, j \in \{0, 1\}$; G be the *Grassmann algebra* on a countable set of anticommuting generators $\{e_1, e_2, \dots \mid e_i e_j = -e_j e_i\}$ with the natural \mathbb{Z}_2 -grading (G_0 and G_1 are spanned by the words of even and, respectively, odd length on the generators $\{e_i\}$). The *Grassmann envelope* $G(\mathcal{A})$ of the superalgebra \mathcal{A} is the subalgebra $G_0 \otimes \mathcal{A}_0 + G_1 \otimes \mathcal{A}_1$ of the tensor product $G \otimes \mathcal{A}$. Recall [19, 20, 27, 28] that \mathcal{A} is said to be a \mathcal{V} -superalgebra if $G(\mathcal{A}) \in \mathcal{V}$. Consider a free \mathcal{V} -superalgebra $\mathcal{F}_{\mathcal{V}}^{(s)}[U]$ on some finite set $U = \{u_1, \dots, u_r\}$ of free odd generators. Let us fix in its Grassmann envelope the elements $u_{ij} = u_i \otimes e_{i+rj}$, where $i = 1, \dots, r$ and $j = 0, 1, \dots$. Then a subalgebra of $G(\mathcal{F}_{\mathcal{V}}^{(s)}[U])$ generated by all the elements u_{ij} is called the *Grassmann \mathcal{V} -algebra of rank r* and is denoted by $\tilde{G}_r(\mathcal{V})$. We stress that, by definition, the generators of $\tilde{G}_r(\mathcal{V})$ form r distinct countable families $\{u_{i0}, u_{i1}, \dots, u_{in}, \dots \mid i = 1, \dots, r\}$ such that every monomial of $\tilde{G}_r(\mathcal{V})$ is skew-symmetric with respect to its variables that belong to the same family.

In 2005, S. V. Pchelintsev [16, 17] studied the identities of Grassmann RA_2 -algebras of ranks 1 and 2. In particular, a finite basis for identities of $\tilde{G}_1(\text{RA}_2)$ was constructed and the Specht property of the variety $\text{Var } \tilde{G}_1(\text{RA}_2)$ generated by $\tilde{G}_1(\text{RA}_2)$ was proved. Moreover, it was shown that $\text{Var } \tilde{G}_2(\text{RA}_2)$ is not Spechtian.

In view of the referred results, the following question gives rise: whether for every finitely based variety \mathcal{V} its Grassmann algebra $\tilde{G}_r(\mathcal{V})$ has a finite basis for identities? In the present paper, we give the negative answer to this question.

Let us fix a field \mathcal{F} of characteristic $\text{char}(\mathcal{F}) \neq 2, 3$ and consider the *subvariety* \mathcal{V} of RA'_2 distinguished by the identity

$$[(x, yz, x), t] = 0, \quad (4)$$

where $[a, b] = ab - ba$ is the commutator of the elements a, b . We prove the following

Theorem. *The variety $\text{Var } \tilde{G}_2(\mathcal{V})$ is a non-finitely based subvariety of \mathcal{V} distinguished by the system of identities*

$$\left(x, \left(y_1, \dots, (y_{n-1}, (y_n, x, y_n), y_{n+1}), \dots, y_{2n-1} \right), x \right) = 0, \quad n = 1, 2, \dots \quad (5)$$

Moreover, $\text{Var } \tilde{G}_2(\mathcal{V})$ can be generated by the Grassmann envelope of a five-dimensional superalgebra on two odd generators with one-dimensional even part.

1 Linear generators of the free algebra $\mathcal{F}_{\mathcal{V}}[X]$

1.1 Common notations

Throughout the paper, all the vector spaces (algebras, superalgebras) are considered over the field \mathcal{F} . Let us fix the following notations:

$\text{rest}(n, m)$ is a rest of the integer division of n by m ;

L_a and R_a are operators of left and right multiplication by the element a , respectively;

T_a is the common denotation for L_a and R_a ;

$\mathcal{M}(A)$ is an algebra of multiplications of an algebra A , i. e. an associative algebra that is generated by all the operators T_a ($a \in A$) and by the identical mapping id ;

$\mathcal{M}(A)'$ is an algebra generated by the restrictions of all operators from $\mathcal{M}(A)$ on A^2 ;

$X = \{x_1, x_2, \dots\}$ is a fixed countable set and $X_d = \{x_1, x_2, \dots, x_d\}$;

$\mathcal{F}_{\mathcal{V}}[Y]$ is a free algebra of a variety \mathcal{V} on a set Y of free generators over \mathcal{F} ;

$T_i = T_{x_i}$ is an operator of $\mathcal{M}(\mathcal{F}_{\mathcal{V}}[X])$;

$\mathcal{P}_d(\mathcal{V})$ is a subspace of all multilinear polynomials of degree $d \geq 3$ in $\mathcal{F}_{\mathcal{V}}[X_d]$;

S_n is a symmetric group on the set $1, 2, \dots, n$;

C_n is a subgroup of S_n generated by the cycle $(12\dots n)$.

Let $f = f(x_1, \dots, x_n)$ be a polynomial of $\mathcal{F}_{\mathcal{V}}[X]$ that is linear with respect to some its variables x_{i_1}, \dots, x_{i_k} , $k \geq 2$. Then we set

$$f(x_1, \dots, \check{x}_{i_1}, \dots, \check{x}_{i_k}, \dots, x_n) = \sum_{\sigma \in S_k} f(x_1, \dots, x_{i_{\sigma(1)}}, \dots, x_{i_{\sigma(k)}}, \dots, x_n),$$

where the permutations are realized with respect to the variables x_{i_1}, \dots, x_{i_k} . The symbol \sim indicates the variables taking part in the permutations. Similarly,

$$f(x_1, \dots, \bar{x}_{i_1}, \dots, \bar{x}_{i_k}, \dots, x_n) = \sum_{\sigma \in C_k} f(x_1, \dots, x_{i_{\sigma(1)}}, \dots, x_{i_{\sigma(k)}}, \dots, x_n).$$

While writing down operators of $\mathcal{M}(\mathcal{F}_V[X])$ we mark naturals with the symbols $\bar{}$ and \sim assuming that these symbols are arranged over the variables with the indices equal to the marked naturals.

Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a superalgebra. It is well known that $G(\mathcal{A})$ satisfies a multilinear identity $f = 0$ iff \mathcal{A} satisfies the graded identity $\tilde{f} = 0$ called the *superization of $f = 0$* . Here, \tilde{f} denotes the so-called *superpolynomial corresponding to f* and we say that \mathcal{A} satisfies the *superidentity $\tilde{f} = 0$* . The detailed descriptions of the process of constructing of superpolynomials (the *superizing process*) can be found in [19, 20, 27, 28].

1.2 Operator relations

Lemma 1.1. *A right alternative algebra A satisfies identity (3) if and only if the operator $L_a R_b$ is skew-symmetric with respect to a, b in $\mathcal{M}(A)$.*

Proof. First assume that $L_a R_b = -L_b R_a$ in $\mathcal{M}(A)$. Then using (1), we have

$$a^2 \circ b = a^2 b + b a^2 = a^2 b + (b a) a = a(L_a R_b + L_b R_a) = 0.$$

By linearization of the obtained equality we get (3).

Conversely, combining (1) with its linearization and (3), we obtain

$$2b L_a R_a = (ab) a - a^2 b + a(b \circ a) = (ab) a + b a^2 - (a \circ b) a = b a^2 - (ba) a = 0. \quad \square$$

We set $\mathfrak{A} = \mathcal{F}_V[X]$. Lemma 1.1 yields that $\mathcal{M}(\mathfrak{A})$ satisfies the relation

$$L_x R_x = 0. \tag{6}$$

Proposition 1.1. *The algebra $\mathcal{M}(\mathfrak{A})'$ satisfies the relations*

$$R_x R_y = 0, \tag{7}$$

$$L_x L_y = [L_y, R_x], \tag{8}$$

$$R_x L_x R_y = 0. \tag{9}$$

Proof. Suppose $w \in \mathfrak{A}^2$. Using (2) and (6), we have

$$w R_x R_y = x L_w R_y = -x L_y R_w = -(yx) w = 0.$$

Applying the linearization of (1) with (2), we obtain

$$w L_x L_y = -(y, x, w) = (y, w, x) = w [L_y, R_x].$$

Combining (2), (4), (6), (7), and (8), we calculate:

$$\begin{aligned} 2w R_x L_x R_y &= 2w [R_x, L_x] R_y = -2(x, w, x) y = -(x, w, x) \circ y - [(x, w, x), y] = \\ &= (x(wx)) (R_y + L_y) = (wx) (L_x R_y + L_x L_y) = (wx) (L_x R_y + [L_y, R_x]) = \\ &= (wx) (L_x R_y + L_y R_x - R_x L_y) = -(wx) R_x L_y = -w R_x^2 L_y = 0. \quad \square \end{aligned}$$

1.3 Standard operators

A *standard operator* is the identical mapping or an operator $H \in \mathcal{M}(\mathfrak{A})$ of the form

$$H = R_{j_0}^\varepsilon L_{j_1} R_{j_2} \dots L_{j_{2k-1}} R_{j_{2k}} L_{j_{2k+1}}^{\varepsilon'},$$

where $\varepsilon, \varepsilon' \in \{0, 1\}$ and $T_x^\varepsilon = \begin{cases} \text{id}, & \text{if } \varepsilon = 0, \\ T_x, & \text{if } \varepsilon = 1; \end{cases}$ the pair $(\varepsilon, \varepsilon')$ is called the *type of H* and is denoted by $\tau(H)$.

We stress that applying relations (7) and (8) it is not hard to prove that the following lemma holds.

Lemma 1.2. *The algebra $\mathcal{M}(\mathfrak{A})$ is a linear span of standard operators.*

Furthermore, the following lemma is an immediate consequence of relations (6) and (9).

Lemma 1.3. *A standard operator of type $(0, 0)$ is skew-symmetric with respect to all its variables.*

1.4 Standard monomials

Let w be a monomial in \mathfrak{A} . Then w is called the *standard monomial of type $\tau(w)$* if $w = x_i H$, where H is a standard operator distinct from R_j and $\tau(w) = \tau(H)$. Note that, by definition, the elements of X^2 are standard monomials of type $(0, 1)$. Further, an *origin of w* is a monomial $w_0 = x_i R_j^\varepsilon$, where $\tau(w) = (\varepsilon, \varepsilon')$; a *formative operator of w* is an operator $F(w)$ such that $w = w_0 F(w) L_k^{\varepsilon'}$.

The following lemma is an immediate consequence of Lemma 1.2.

Lemma 1.4. *The algebra \mathfrak{A} is spanned by the standard monomials.*

We say that a standard monomial w is *nondegenerate* if $F(w) \neq \text{id}$. Otherwise, w is called *degenerate*.

Lemma 1.5. *Every nondegenerate standard monomial w satisfies the following conditions.*

1. *The formative operator $F(w)$ is skew-symmetric with respect to all its variables.*
2. *The origin w_0 is skew-symmetric with respect to its variables.*

Proof. The first condition follows from Lemma 1.3. Using (1) and (7), we prove the second one:

$$y^2 L_x R_z = (xy^2) z = (xy) R_y R_z = 0. \quad \square$$

1.5 Basis monomials

A *basis monomial* is a standard monomial w such that the sequences of indices of the variables of the origin w_0 and of the formative operator $F(w)$ ascend strictly. In particular, all the standard monomials of degrees 1 and 2 are basis ones.

Lemma 1.6. *The algebra \mathfrak{A} is spanned by the basis monomials.*

Proof. By virtue of lemma 1.4, it suffices to prove that every standard monomial of degree not less than 3 can be represented as a linear combination of basis monomials. Let us rewrite down (1) in the form $y^2 L_x = y L_x R_y$. Then it is clear that an origin of a degenerate standard monomial of degree 3 is skew-symmetric modulo linear combinations of nondegenerate standard monomials. Hence, to conclude the proof it remains to note that by lemma 1.5, every nondegenerate standard monomial is proportional to a basis one. \square

A *basis polynomial* is a linear combination over \mathcal{F} of pairwise distinct basis monomials with nonzero scalars.

Lemma 1.7. *Every T-ideal of \mathfrak{A} can be generated by a system of multilinear basis polynomials.*

Proof. Note that every basis monomial by its definition has a degree not more than 3 with respect to any of its variables. Consequently, in view of the restrictions $\text{char}(\mathcal{F}) \neq 2, 3$, a T-ideal of \mathfrak{A} generated by some basis polynomial can be also generated by a system of multilinear polynomials (see [29, Chap. 1]). Moreover, by Lemma 1.6, this polynomials can be expressed linearly with multilinear basis polynomials. \square

2 Auxiliary \mathcal{V} -superalgebra

Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a superalgebra

$$\mathcal{A}_0 = \mathcal{F} \cdot a, \quad \mathcal{A}_1 = \mathcal{F} \cdot v + \mathcal{F} \cdot w + \mathcal{F} \cdot y + \mathcal{F} \cdot z + \mathcal{F} \cdot z'$$

such that all nonzero products of its basis elements are the following:

$$z \cdot z = a, \quad y \cdot a = v, \quad z' \cdot a = w, \quad y \cdot v = v \cdot y = a.$$

By definition, it is not hard to see that \mathcal{A} is a metabelian superalgebra generated by the odd elements y, z, z' and w lies in the annihilator of \mathcal{A} . Moreover, the direct verification shows that the following proposition holds.

Proposition 2.1. *The superalgebra \mathcal{A} satisfies the relations*

$$\mathcal{A}_0 \mathcal{A} = 0, \tag{10}$$

$$[\mathcal{A}_1, \mathcal{A}_1] = 0, \tag{11}$$

$$\mathcal{A}_0 [L_{u_i}, L_{u_j}] = 0, \tag{12}$$

$$\mathcal{A}_1 \mathcal{A}_0 [L_{u_i}, L_{u_j}] L_{u_k} = 0, \tag{13}$$

where u_i, u_j, u_k are arbitrary generators of \mathcal{A} .

We stress that relation (10) yields the following

Lemma 2.1. *The operator $L_a R_b$ annihilates \mathcal{A}_1 for all $a, b \in \mathcal{A}$.*

Lemma 2.2. *Every metabelian superalgebra \mathcal{A} generated by odd elements and satisfying (10)–(13) is a \mathcal{V} -superalgebra.*

Proof. Throughout the proof, the conditions of the metability of \mathcal{A} , the odd parity of all generators of \mathcal{A} , and its consequence $\mathcal{A}_0 \subset \mathcal{A}^2$, are used with no comments.

First, let us prove that $G(\mathcal{A})$ satisfies (1), i. e.

$$(a, b, c) + (-1)^{|b||c|} (a, c, b) = 0,$$

for all basis elements $a, b, c \in \mathcal{A}$, where $|a|$ denotes the parity of the element a ($|a| = i \in \{0, 1\}$ if $a \in \mathcal{A}_i$). Note that if $(a, b, c) \neq 0$, then at least two of the elements a, b, c are odd. If $b, c \in \mathcal{A}_1$, then by (10), (11), the associator $(a, b, c) = a(bc)$ is symmetric with respect to b, c . Hence, it remains to check the skew-symmetry of (a, b, c) with respect to b, c under the conditions: $b \in \mathcal{A}_0$ and a, c are generators of \mathcal{A} . In this case, using (10), (11), and (12), we obtain

$$(u_i, b, u_j) = (u_i b) u_j = u_j (u_i b) = u_i (u_j b) = - (u_i, u_j, b).$$

Thus, $G(\mathcal{A})$ is right alternative.

Now, let us prove that $G(\mathcal{A})$ satisfies (3). By Lemma 1.1, it suffices to verify that the operator $L_a R_b$ is skew-symmetric in $\mathcal{M}(G(\mathcal{A}))$. Taking into account Lemma 2.1, it remains to check that

$$\mathcal{A}_0 (L_{u_i} R_{u_j} - L_{u_j} R_{u_i}) = 0.$$

Indeed, assuming $b \in \mathcal{A}_0$ and applying (11), (12), we have

$$(u_i b) u_j = u_j (u_i b) = u_i (u_j b) = (u_j b) u_i.$$

Finally, let us prove that $G(\mathcal{A})$ satisfies (4). In view of (1) and (2) it suffices to verify that $L_x L_x R_t = L_x L_x L_t$ in $\mathcal{M}(G(\mathcal{A}))'$, i. e.

$$b [L_{u_i}, L_{u_j}] R_{u_k} = (-1)^{|b|} b [L_{u_i}, L_{u_j}] L_{u_k}, \quad b \in \mathcal{A}^2.$$

If $b \in \mathcal{A}_0$, then the left side of the equality is zero by virtue of (10) and the right side is zero in view of (12). Otherwise, if $b \in \mathcal{A}_1 \cap \mathcal{A}^2 = \mathcal{A}_1 \mathcal{A}_0$, then the both sides of the equality are zeros by virtue of (11) and (13). \square

Lemma 2.3. *A value taken by an arbitrary nondegenerate basis monomial w of type $(\varepsilon, \varepsilon')$ on basis elements of \mathcal{A} lies in the homogeneous component $\mathcal{A}_{\varepsilon'}$.*

Proof. Let $0 \neq \tilde{w} \in \mathcal{A}$ be a monomial obtained by a substitution of the variables of w by arbitrary basis elements of \mathcal{A} . Then Lemma 2.1 implies $\tilde{w}_0 \in \mathcal{A}_0$. Hence, by virtue of the metability and the odd parity of the generators of \mathcal{A} , we have $\tilde{w}_0 F(\tilde{w}) \in \mathcal{A}_0$. Therefore, $\tilde{w} \in \mathcal{A}_{\varepsilon'}$. \square

3 Additive basis of the space $\mathcal{P}_d(\mathcal{V})$

A *regular polynomial* is a basis polynomial $f \in \mathcal{P}_d(\mathcal{V})$ represented as a linear combination of nondegenerate basis monomials of a same fixed type. This type is called the *type of f* and is denoted by $\tau(f)$. If $f = 0$ is an identity of some algebra $A \in \mathcal{V}$, then we say that A satisfies a *regular identity of type $\tau(f)$* .

3.1 Reduction to the regular identities of $G(\mathcal{A})$

Let \mathcal{A} be the \mathcal{V} -superalgebra defined in Sec. 2.

Lemma 3.1. *If $G(\mathcal{A})$ satisfies a nontrivial in \mathcal{V} multilinear identity of degree $d \geq 3$, then $G(\mathcal{A})$ satisfies some regular identity.*

Proof. By virtue of Lemma 1.6 we may assume that $G(\mathcal{A})$ satisfies an identity $f = 0$, where $f \in \mathcal{P}_d(\mathcal{V})$ is a linear combination of pairwise distinct basis monomials with nonzero scalars.

First, consider the case $d = 3$. Suppose that f is not regular of type $(0, 0)$, i. e. f contains degenerate basis monomials. Then, in view of (7), fR_4 will be regular of type $(1, 0)$.

In the case $d \geq 4$, if f is not regular, then it can be represented as a sum $f_0 + f_1$ of two regular polynomials of types $\tau(f_0) = (1 - \varepsilon, 0)$ and $\tau(f_1) = (\varepsilon, 1)$, where $\varepsilon = \text{rest}(d, 2)$. By Lemma 2.3, all the values taken by $\tilde{f}_i(0, 1)$ on basis elements of \mathcal{A} lie in \mathcal{A}_i . This yields that $G(\mathcal{A})$ satisfies both identities $f_i = 0$. \square

3.2 Reduction to the regular identities of $G(\mathcal{A})$ of type $(\varepsilon, 0)$

For $k = 1, \dots, d + 1$ we define linear mappings

$$L_k^* : \mathcal{P}_d(\mathcal{V}) \mapsto \mathcal{P}_{d+1}(\mathcal{V})$$

acting on the monomials $w = w(x_1, \dots, x_d) \in \mathcal{P}_d(\mathcal{V})$ as follows:

$$wL_k^* = w(x_{1^{(k)}}, \dots, x_{d^{(k)}})L_k, \quad i^{(k)} = \begin{cases} i, & i < k, \\ i + 1, & i \geq k. \end{cases}$$

Lemma 3.2. *If $G(\mathcal{A})$ satisfies a regular identity of type $(\varepsilon, 1)$, then $G(\mathcal{A})$ satisfies a regular identity of type $(\varepsilon, 0)$.*

Proof. Let g be a regular polynomial of $\mathcal{P}_{d+1}(\mathcal{V})$ of type $(\varepsilon, 1)$. Then by the definition of standard monomials we can represent g in the form

$$g = \sum_{k \in I} f_k L_k^*,$$

where $\emptyset \neq I \subseteq \{1, \dots, d + 1\}$ and every $f_k \in \mathcal{P}_d(\mathcal{V})$ is a regular polynomial of type $(\varepsilon, 0)$.

Let us prove that the identity $g = 0$ in $G(\mathcal{A})$ implies $f_k = 0$. Indeed, assume that $\tilde{g} = 0$ in \mathcal{A} and \tilde{f}_k takes a nonzero value at some basis elements $x_1 = b_1, \dots, x_d = b_d$

of \mathcal{A} . Then Lemma 2.3 implies $\tilde{f}_k \in \mathcal{A}_0$ and, consequently, by the definition of \mathcal{A} , we have $\tilde{f}_k = \alpha a$, where $0 \neq \alpha \in \mathcal{F}$. But in this case, \tilde{g} takes a nonzero value in \mathcal{A} at the elements $x_{1\langle k \rangle} = b_1, \dots, x_{d\langle k \rangle} = b_d, x_k = z'$:

$$\tilde{g} = \tilde{f}_k L_{z'} = \alpha z' \cdot a = \alpha w \neq 0.$$

The obtained contradiction completes the proof. \square

3.3 Linear independence of the basic monomials in $\mathcal{P}_d(\mathcal{V})$

In order to avoid complicated formulas while writing down the elements of $\mathcal{P}_d(\mathcal{V})$ we omit the indices of variables at the operator symbols L, R and assume that they are arranged in the ascending order. For example, the notation $(x_2 x_5)(LR)^2$ means the monomial $(x_2 x_5) L_1 R_3 L_4 R_6$.

Lemma 3.3. *The set of all basis monomials in $\mathcal{P}_d(\mathcal{V})$ forms its additive basis.*

Proof. By virtue of Lemmas 1.6, 2.2, 3.1 and 3.2, it suffices to prove that $G(\mathcal{A})$ does not satisfy any regular identity of type $(\varepsilon, 0)$. Consider a regular polynomial $f \in \mathcal{P}_d(\mathcal{V})$ of type $\tau(f) = (\varepsilon, 0)$. If d is even, then we can represent f in the form

$$f = \sum_{i < j} \alpha_{i,j} (x_i x_j) (LR)^{\frac{d}{2}-1}, \quad 0 \neq \alpha_{i,j} \in \mathcal{F}.$$

By $\tilde{f}_{i,j}$ denote the value taken by the superpolynomial \tilde{f} on the following elements of \mathcal{A} :

$$x_i = x_j = z, \quad x_k = y \text{ for all } k \neq i, j.$$

Hence, $\tilde{f}_{i,j}$ is proportional to the element

$$(z \cdot z)(L_y R_y)^{\frac{d}{2}-1} = a \neq 0.$$

For odd d , we have

$$f = \sum_i \alpha_i x_i (LR)^{\frac{d-1}{2}}, \quad 0 \neq \alpha_i \in \mathcal{F}.$$

Similarly, a value \tilde{f}_i taken by \tilde{f} on the elements

$$x_i = a, \quad x_j = y \text{ for all } j \neq i$$

turns out to be proportional to

$$a(L_y R_y)^{\frac{d-1}{2}} = a \neq 0. \quad \square$$

We stress that, in view of Lemma 1.7, the proof of Lemma 3.3 implies $\mathcal{V} = \text{Var } \tilde{G}_3(\mathcal{V})$.

4 Auxiliary polynomials

4.1 Polynomials ξ, ψ, ϕ

Consider the following polynomials in \mathfrak{A} :

$$\begin{aligned}\xi(x, y, z, t) &= (xy) L_z R_t + (zt) L_x R_y, \\ \psi(x, y, z, t) &= (\bar{x}\bar{y}) L_{\bar{z}} R_t, \quad \phi(x, y, z, t) = (\bar{x}\bar{y}) L_{\bar{z}} R_{\bar{t}}.\end{aligned}$$

Lemma 1.5 yields immediately the following properties of ξ and ψ .

Lemma 4.1. *The polynomial $\xi(x, y, z, t)$ is skew-symmetric w.r.t. the pair x, y and, independently, w.r.t. the pair z, t .*

Lemma 4.2. *The polynomial $\psi(x, y, z, t)$ is skew-symmetric w.r.t. x, y, z .*

Moreover, combining the definition of ξ with Lemma 4.1, we obtain the following

Lemma 4.3. *The polynomial ξ is invariant under the action of the Klein four-group on its variables:*

$$\xi(x, y, z, t) = \xi(y, x, t, z) = \xi(t, z, y, x).$$

Proposition 4.1. *The algebra \mathfrak{A} satisfies the identities*

$$\phi(ab, x, y, z) = 0, \tag{14}$$

$$\phi(x, y, z, t) = \xi(\bar{x}, y, \bar{z}, t) = \xi(x, \bar{y}, z, \bar{t}), \tag{15}$$

$$\psi(a, x, b, x) + \frac{1}{2}\phi(a, x, b, x) = 2(ax) L_b R_x, \tag{16}$$

$$\phi(a, \bar{x}, \bar{y}, \bar{z}) = 0. \tag{17}$$

Proof. Applying (2), (7), (8), and Lemma 1.3, we have

$$\phi(ab, x, y, z) = (ab) (R_x L_y R_z + L_z L_x R_y) = (ab) (R_x L_y R_z - R_z L_x R_y) = 0.$$

To prove (15), first note that the equality $\xi(\bar{x}, y, \bar{z}, t) = \xi(x, \bar{y}, z, \bar{t})$ follows from the definition of ξ . Then we stress that ϕ can be represented, by definition, as follows:

$$\phi(x, y, z, t) = \xi(x, y, z, t) + \xi(t, x, y, z).$$

Therefore, applying Lemma 4.3 to the second summand, we obtain (15).

Further, combining Lemma 1.5 with (15), we prove (16):

$$\psi(a, x, b, x) + \frac{1}{2}\phi(a, x, b, x) = (ax) L_b R_x + (xb) L_a R_x + \xi(a, x, b, x) = 2(ax) L_b R_x.$$

Finally, using (15) and taking into account Lemma 4.1, we get

$$\phi(a, \bar{x}, \bar{y}, \bar{z}) = \xi(a, \bar{x}, y, \bar{z}) + \xi(a, \bar{y}, z, \bar{x}) + \xi(a, \bar{z}, x, \bar{y}) = \xi(a, \check{x}, \check{y}, \check{z}) = 0. \quad \square$$

The following lemma is an immediate consequence of (15).

Lemma 4.4. *The polynomial $\phi(x, y, z, t)$ is symmetric w.r.t. the pair x, z and, independently, w.r.t. the pair y, t .*

4.2 Special regular polynomials

By definition, the space $\mathcal{P}_d(\mathcal{V})$ can be divided into two components

$$\mathcal{P}_d(\mathcal{V}) = \mathcal{P}_d^{(0,1-\varepsilon)}(\mathcal{V}) + \mathcal{P}_d^{(1,\varepsilon)}(\mathcal{V}), \quad \varepsilon = \text{rest}(d, 2),$$

where $\mathcal{P}_d^{(\varepsilon,\varepsilon')}$ denotes the subspace of all regular polynomials in $\mathcal{P}_d(\mathcal{V})$ of type $(\varepsilon, \varepsilon')$.

Let us define some special polynomials in $\mathcal{P}_d^{(1,1)}(\mathcal{V})$ for $d = 2n + 3$, $n \in \mathbb{N}$.

We use ϑ as a common denotation for the symbols ξ, ψ, ϕ . The polynomial $\vartheta(x_i, x_j, x_k, x_\ell)$ is denoted shortly by $\vartheta(i, j, k, \ell)$.

A ϑ -word of order n is a polynomial $f \in \mathcal{P}_d^{(1,1)}(\mathcal{V})$ of the form

$$f = \vartheta(i, j, k, \ell) (LR)^{n-1} L_m$$

denoted by $f = \vartheta_n(i, j, k, \ell, m)$. The polynomial $f_0 = \vartheta(i, j, k, \ell)$ is called the *origin* of f .

A double ϕ -word of order $n \geq 2$ is a polynomial $f \in \mathcal{P}_d^{(1,1)}(\mathcal{V})$ ($d \geq 7$) of the form

$$f = \phi(i, j, k, \ell) L_{\bar{m}} R (LR)^{n-2} L_{\bar{q}}$$

denoted by $f = \varphi_n(i, j, k, \ell, m, q)$. The polynomial $f_0 = \phi(i, j, k, \ell)$ is called the *origin* of f . Note that in view of Lemma 1.3, f can be represented as a linear combination of two ϕ -words of order n with the same origins f_0 .

A triple ϕ -word of order n is a polynomial in $\mathcal{P}_d^{(1,1)}(\mathcal{V})$ of the form $\phi_n(\bar{i}, j, \bar{k}, \ell, \bar{m})$.

Proposition 4.2. *The following identity holds for $n \geq 2$:*

$$\varphi_n(1, 2, 3, \bar{4}, \bar{5}, \bar{6}) = -\phi_n(\bar{4}, 1, \bar{5}, 3, \bar{6}). \quad (18)$$

Proof. By Lemma (1.5), we have

$$\begin{aligned} \phi(a, b, c, x) L_x R_y &= (xa) L_b R_c L_x R_y + (cx) L_a R_b L_x R_y = \\ &= -(xa) L_x R_c L_b R_y - (xc) L_x R_a L_b R_y = -\xi(x, a, x, c) L_b R_y. \end{aligned}$$

Multiplying the both sides of the obtained equality by HL_x , where

$$H = \begin{cases} \text{id}, & \text{if } n = 2, \\ L_8 R_9 \dots L_{2n+2} R_{2n+3}, & \text{if } n \geq 3, \end{cases}$$

we get

$$\phi(a, b, c, x) L_x R_y H L_x = -\xi(x, a, x, c) L_b R_y H L_x.$$

Hence, by the linearization

$$a = x_1, \quad b = x_2, \quad c = x_3, \quad x \mapsto x_4, x_5, x_6, \quad y = x_7,$$

taking into account (15), we obtain

$$\varphi_n(1, 2, 3, \bar{4}, \bar{5}, \bar{6}) = -\xi_n(\bar{4}, 1, \bar{5}, 3, \bar{6}) = -\phi_n(\bar{4}, 1, \bar{5}, 3, \bar{6}). \quad \square$$

Let us set

$$g_n = g_n(x, y_1, \dots, y_{2n-1}) = \left(x, \left(y_1, \dots, (y_{n-1}, (y_n, x, y_n), y_{n+1}), \dots, y_{2n-1} \right), x \right).$$

Lemma 4.5. *A linearization of g_n in $\mathcal{P}_d^{(1,1)}(\mathcal{V})$ is proportional to a triple ϕ -word of order n .*

Proof. Applying Lemma 1.3 and relation (7), we have

$$\begin{aligned} g_n(x, z_{2n-2}, \dots, z_2, y, z_1, z_3, \dots, z_{2n-3}) &= \\ &= x [L_y, R_y] [L_{z_2}, R_{z_1}] \dots [L_{z_{2n-2}}, R_{z_{2n-3}}] [L_x, R_x] = \\ &= x R_y L_y [L_{z_2}, R_{z_1}] \dots [L_{z_{2n-2}}, R_{z_{2n-3}}] R_x L_x = \\ &= (-1)^{n-1} (xy) L_y R_{z_1} L_{z_2} \dots R_{z_{2n-3}} L_{z_{2n-2}} R_x L_x = \\ &= (-1)^n (xy) L_x R_y L_{z_1} R_{z_2} \dots L_{z_{2n-3}} R_{z_{2n-2}} L_x. \end{aligned}$$

Hence, linearizing $g_n \mapsto \Delta g_n \in \mathcal{P}_d^{(1,1)}(\mathcal{V})$:

$$x \mapsto x_1, x_2, x_3, \quad y \mapsto x_4, x_5, \quad z_1 = x_6, \dots, z_{2n-2} = x_d,$$

and taking into account (15), we obtain

$$\Delta g_n = (-1)^n \xi_n(\check{1}, 4, \check{2}, 5, \check{3}) = (-1)^n \phi_n(\bar{1}, 4, \bar{2}, 5, \bar{3}). \quad \square \quad (19)$$

Lemma 4.6. *The intersection $\mathcal{I} = (g_n)^T \cap \left(\bigcup_{d=2n+3}^{\infty} \mathcal{P}_d(\mathcal{V}) \right)$ is spanned by the linearizations of g_n .*

Proof. Taking into account Lemma 4.5, it remains to prove that

$$(\Delta g_n)^T \cap \left(\bigcup_{d=2n+4}^{\infty} \mathcal{P}_d(\mathcal{V}) \right) = \{0\}.$$

In view of (2) and (14), it suffices to verify that a triple ϕ -word of any order lies in the annihilator $\text{Ann } \mathfrak{A}$. Following the proof of Lemma 4.5, we may restrict with checking $h \in \text{Ann } \mathfrak{A}$ for the monomial

$$h = (xy) L_x R_y (LR)^{n-1} L_x.$$

Indeed, Lemma 1.3 yields immediately $h R_z = 0$ and, using (7) and (8), we get

$$h L_z = (xy) L_x R_y (LR)^{n-1} L_x L_z = (xy) L_x R_y (LR)^{n-1} L_z R_x = 0. \quad \square$$

5 Linear generators of the space $\mathcal{P}_d(\mathfrak{M})$

Let \mathfrak{M} be a subvariety of \mathcal{V} distinguished by system (5). In what follows, considering the free algebra $\mathfrak{A}' = \mathcal{F}_{\mathfrak{M}}[X]$ and its subspace $\mathcal{P}_d(\mathfrak{M})$, we assume that they inherit naturally all the notions introduced for \mathfrak{A} and $\mathcal{P}_d(\mathcal{V})$ in previous sections.

5.1 Normalized words

Let Φ_d be a linear span of all ϕ -words in $\mathcal{P}_d^{(1,1)}(\mathfrak{M})$ for $d = 2n+3$, $n \in \mathbb{N}$. A ϕ -word $f \in \Phi_d$ of the form

$$f = \phi(i, j, k, \ell) (LR)^{n-1} L_m$$

is called *normalized* if m is a maximal index in the operator $(LR)^{n-1} L_m$. In particular, every ϕ -word of order 1 is normalized. We denote a normalized ϕ -word shortly, omitting the corresponding minimal index:

$$f = \phi_n(i, j, k, \ell).$$

In view of Lemma 1.3, the given definition implies instantly the following

Lemma 5.1. *Every ϕ -word f is either normalized or can be expressed linearly with a normalized ϕ -word and a double ϕ -word with the same origins f_0 .*

Let Φ'_d , for $d \geq 7$, be a subspace of Φ_d generated by all double ϕ -words. A double ϕ -word $f \in \Phi'_d$ of the form

$$f = \phi(i, j, k, \ell) L_{\bar{m}} R (LR)^{n-2} L_{\bar{q}}, \quad n \geq 2$$

is called *normalized* if m is a minimal index in the operator $L_m R (LR)^{n-1}$. We denote f shortly, omitting m :

$$f = \varphi_n(i, j, k, \ell, q).$$

Lemma 5.2. *Every double ϕ -word $f = \varphi_n(i, j, k, \ell, m, q)$ is either normalized or can be expressed linearly with two normalized double ϕ -words with the same origins f_0 .*

Proof. Let m' be a minimum of the set $\{1, \dots, d\} \setminus \{i, j, k, \ell\}$. By assumption of the lemma it is clear that $m' < m$ and $m' < q$. We stress that in view of Lemma 1.3, f satisfies the assertion of Lemma 5.3 for $x_m, x_q, x_{m'}$. Consequently, we can represent f in the form

$$f = \pm \varphi_n(i, j, k, \ell, m) \pm \varphi_n(i, j, k, \ell, q). \quad \square$$

We call the procedure described in Lemma 5.2 the *normalization of double ϕ -word*.

5.2 Tame words

Let $f = f(x, y, \dots, z)$ be a nonassociative multilinear polynomial and $S\{f\}$ be the symmetric group on the set x, y, \dots, z . For $\sigma \in S\{f\}$ we set

$$f^\sigma = f(\sigma(x), \sigma(y), \dots, \sigma(z)).$$

The key part in the proof of the statements of this subsection is played by the following obvious lemma.

Lemma 5.3. *If $f = f(x, y, z_1, \dots, z_n)$ is symmetric w.r.t. x, y and*

$$f(\bar{x}, \bar{y}, \bar{z}_1, z_2, \dots, z_n) = f(\bar{x}, \bar{y}, z_1, \bar{z}_2, \dots, z_n) = f(\bar{x}, \bar{y}, z_1, z_2, \dots, \bar{z}_n) = 0,$$

then the vector space $\text{Vec}_{\mathcal{F}} \langle f^\sigma \mid \sigma \in S\{f\} \rangle$ is spanned by the elements f^σ such that $\sigma(x) = x$.

A ϕ -word $f \in \Phi_d$ is called *tame* if its origin f_0 has one of the following types:

$$1) \phi(1, i, j, d), \quad i < j; \quad 2) \phi(1, 2, d, j).$$

Otherwise, f is called *wild*.

Lemma 5.4. *The space Φ_d is spanned by the tame ϕ -words.*

Proof. Let us show that every wild ϕ -word $f \in \Phi_d$ can be expressed linearly with tame ϕ -words. We do it at four steps, referring each time Lemma 5.3 and using Lemmas 1.3, 4.4 with no comments.

First let us show that f is a linear combination of ϕ -words with origins of the form $\phi(1, i, j, k)$. Indeed, as far as ϕ is cyclic, it suffices to assume that f_0 doesn't contain the variable x_1 . In this case, taking into account that every triple ϕ -word, in view of Lemma 4.6, is zero in $\mathcal{P}_d(\mathfrak{M})$, we see that the assertion of Lemma 5.3 holds for the variables on the first and the third positions in f_0 and for all the variables outside f_0 . Thus applying Lemma 5.3 under the assumption $x = x_1$, we represent f in the required form.

Further, by the similar arguments for the variables on the second and the fourth positions in f_0 , one can show that if f_0 doesn't contain x_d , then f is a linear combination of ϕ -words with origins of the form $\phi(1, i, j, d)$.

Now suppose that f is wild and $f_0 = \phi(1, i, d, j)$. Then applying Lemma 5.3 under the assumption $x = x_2$, we can express f with two tame ϕ -words with the origins of type 2).

Finally, consider the case $f_0 = \phi(1, j, i, d)$, where $j > i$. Using (17) and, if necessary, Lemma 5.3, we express f with one ϕ -word with the origin of type 1) and one (in the case $i = 2$) or two ϕ -words with the origins of type 2). \square

A double ϕ -word in Φ'_d is called *tame* if it is normalized and has one of the following types:

$$1) \varphi_n(1, i, j, d, k), \quad i < j < k; \quad 2) \varphi_n(1, 2, d, j, k), \quad j < k.$$

Otherwise, f is called *wild*.

Lemma 5.5. *The space Φ'_d is spanned by the tame double ϕ -words.*

Proof. The proof is in seven steps.

1. By Lemma 4.6, identity (18) gets in \mathfrak{M} the form

$$\varphi_n(1, 2, 3, \bar{4}, \bar{5}, \bar{6}) = 0.$$

Consequently, in view of the cyclic property of ϕ , a double ϕ -word $\varphi_n(1, 2, 3, 4, 5, 6)$ satisfies the assertion of Lemma 5.3 for the variables

$$x_6 = x, \quad x_5 = y, \quad x_4 = z_1, \dots, \quad x_1 = z_4.$$

Throughout the proof, referring item 1, we apply Lemma 5.3, combining it with Lemmas 1.3 and 4.4 with no comments.

2. Following the procedure of Lemma 5.4, one can prove that every double ϕ -word can be represented as a linear combination of double ϕ -words with the origins of tame ϕ -words.
3. By item 2 and Lemma 5.2, Φ'_d is spanned by the normalized double ϕ -words with the origins of tame ϕ -words.
4. Let $f \in \Phi'_d$ be a wild normalized double ϕ -word with an origin of some tame ϕ -word. Then f has one of the following forms for $i < j < k$:

$$\varphi_n(1, 2, d, k, j), \quad \varphi_n(1, i, k, d, j), \quad \varphi_n(1, j, k, d, i).$$

By item 3, it suffices to prove that f can be expressed linearly with tame double ϕ -words.

5. If $f = \varphi_n(1, 2, d, k, j)$ and $j > m = \min(\{3, \dots, d-1\} \setminus \{k\})$, then by item 1, we have

$$f = -\varphi_n(1, 2, d, j, k) - \varphi_n(1, 2, d, m, j, k).$$

Here, the first summand is tame and the second one is either tame or, after normalization by Lemma 5.2, gets the form

$$\varphi_n(1, 2, d, m, j, k) = \pm \varphi_n(1, 2, d, m, j) \pm \varphi_n(1, 2, d, m, k),$$

where the both summands are tame by the choice of m .

6. If $f = \varphi_n(1, i, k, d, j)$, where $j > m = \min(\{2, \dots, d-1\} \setminus \{i, k\})$, then by item 1, we obtain

$$f = -\varphi_n(1, i, j, d, k) - \varphi_n(1, i, m, d, j, k).$$

Again, the first summand is tame and if $i < m$, then the second summand is either tame or can be normalized, by Lemma 5.2, up to two tame double ϕ -words. Otherwise, for $i > m$, using (17), we get

$$\varphi_n(1, i, m, d, j, k) = -\varphi_n(1, m, i, d, j, k) - \varphi_n(1, m, d, i, j, k).$$

Now, as above, the normalization of the first summand gives two tame double ϕ -words and the second summand, by item 2, can be represented as a linear combination of double ϕ -words with the origins of tame ϕ -words of type 2). Consequently, by Lemma 5.2 and item 5, these double ϕ -words are also linear combinations of tame double ϕ -words.

7. Finally, if $f = \varphi_n(1, j, k, d, i)$, where $i > m = \min(\{2, \dots, d-1\} \setminus \{j, k\})$, then by item 1, we get

$$f = -\varphi_n(1, i, k, d, j) - \varphi_n(1, m, k, d, i, j).$$

The normalization of the second summand and the application of item 6 complete the proof. \square

5.3 Stable basis monomials and basis words

A basis monomial $w \in \mathcal{P}_d(\mathfrak{M})$ is called *stable* if $w \notin \mathcal{P}_d^{(1,1)}(\mathcal{V})$ or the minimal of the indices of the variables of its origin w_0 is less than the minimal index in its formative operator $F(w)$.

Basis words are all elements of $\mathcal{P}_d^{(1,1)}(\mathfrak{M})$ of the following types:

- 1) ψ -words of the form $\psi_n(1^{(i)}, 2^{(i)}, 3^{(i)}, 4^{(i)}, i)$, $1 \leq i \leq d$;
- 2) normalized tame ϕ -words;
- 3) tame double ϕ -words for $d \geq 7$.

Lemma 5.6. *The space $\mathcal{P}_d^{(1,1)}(\mathfrak{M})$ is spanned by its stable basis monomials and basis words.*

Proof. First we stress that by the definition of ψ -word, taking into account Lemma 1.5, we can represent every nonstable basis monomial as a linear combination of two stable basis monomials and a ψ -word of the form $f = \psi_n(1^{(i)}, j, k, \ell, i)$. Then combining Lemmas 1.3 and 4.2 with identity (16), it is not hard to verify that f is skew-symmetric w.r.t. all its variables, except $x_{1^{(i)}}$, modulo Φ_d and linear combinations of stable basis monomials. Consequently, every nonstable basis monomial can be expressed linearly with a stable one and a basis word of type 1) modulo Φ_d . Thus to complete the proof in the case $d = 5$ it suffices to refer Lemma 5.4. Further, for $d \geq 7$, it follows from Lemmas 5.1 and 5.4 that Φ_d is spanned by the basis words of type 2) modulo Φ'_d . Finally, by Lemma 5.5, the basis words of type 3) are linear generators of Φ'_d . \square

6 Proof of the Theorem

First we stress that the polynomial Δg_n of form (19) is regular and, consequently, by Lemma 3.3, $\Delta g_n \neq 0$ in \mathfrak{A} . Thus, \mathfrak{M} is a proper subvariety of \mathcal{V} . Furthermore, by Lemma 4.6, we have $(g_n)^T \cap (g_m)^T = \{0\}$ for $n \neq m$. Therefore, \mathfrak{M} is non-finitely based.

Lemma 6.1. *The Grassmann algebra $\tilde{G}_2(\mathcal{V})$ lies in \mathfrak{M} .*

Proof. Consider the linearization $f = \Delta g_n$ of form (19). By Lemma 4.6, it suffices to verify that $f = 0$ on every set of generators of $\tilde{G}_2(\mathcal{V})$. Indeed, if we substitute the variables of \tilde{f} by the generators of the free \mathcal{V} -superalgebra $\mathcal{F}_{\mathcal{V}}^{(s)}[u_1, u_2]$, then at least two of the variables x_1, x_2, x_3 get the same value. Consequently, in view of the odd parity of u_1, u_2 , the linear combination \tilde{f} contains with every its monomial αw ($\alpha = \pm 1$) the monomial $-\alpha w$. Hence, $\tilde{f} = 0$ in $\mathcal{F}_{\mathcal{V}}^{(s)}[u_1, u_2]$. \square

Let $\mathcal{A}' = \mathcal{A}'_0 \oplus \mathcal{A}'_1$ be a superalgebra

$$\mathcal{A}'_0 = \mathcal{F} \cdot a, \quad \mathcal{A}'_1 = \mathcal{F} \cdot v + \mathcal{F} \cdot w + \mathcal{F} \cdot y + \mathcal{F} \cdot z$$

such that all nonzero products of its basis elements are the following:

$$z \cdot z = a, \quad y \cdot a = v, \quad z \cdot a = w, \quad y \cdot v = v \cdot y = a.$$

The direct verification shows that all the statements of Sec. 2 formulated for \mathcal{A} hold for \mathcal{A}' as well. Moreover, \mathcal{A}' satisfies all the Lemmas of Sec. 3, except Lemma 3.2, that is true only for $\varepsilon = 0$. Consequently, taking into account Lemma 6.1, we obtain that \mathcal{A}' is an \mathfrak{M} -superalgebra such that all the stable basis monomials, not lying in $\mathcal{P}_d^{(1,1)}(\mathfrak{M})$, for $d \geq 5$, are linearly independent on $G(\mathcal{A}')$. Therefore, in view of Lemmas 1.7 and 5.6, to prove the Theorem it suffices to verify the linear independence of the stable basis monomials and the basis words of $\mathcal{P}_d^{(1,1)}(\mathfrak{M})$ on $G(\mathcal{A}')$.

We set, as above, $d = 2n + 3$, $n \in \mathbb{N}$ and also assume $\Phi'_5 = 0$.

Lemma 6.2. *If $f = 0$ in $G(\mathcal{A}')$ for some $f \in \mathcal{P}_d^{(1,1)}(\mathfrak{M})$, then $f \in \Phi_d$.*

Proof. By Lemma 5.6, we can write down f in the form

$$f \equiv \sum_{k=1}^d f_k L_{x_k}^* \pmod{\Phi_d}$$

such that

$$f_k = \sum_{i=2}^{d-1} \alpha_{i,k} (x_1 x_i) (LR)^n + \beta_k \psi(1, 2, 3, 4) (LR)^{n-1},$$

where $\alpha_{i,k}, \beta_k \in \mathcal{F}$. By $\tilde{f}_{i,k}$ denote the value taken by the superpolynomial \tilde{f} on the following elements of \mathcal{A}' :

$$x_{i^{(k)}} = v, \quad x_k = z, \quad x_j = y \quad \text{for } j \in [1, d] \setminus \{i^{(k)}, k\}.$$

While calculating the values $\tilde{f}_{i,k}$, we take into account (14). Suppose $\alpha_{i,k} \neq 0$ for $i \geq 4$; then $\tilde{f}_{i,k}$ is proportional to the element

$$(y \cdot v) (L_y R_y)^n L_z = a (L_y R_y)^n L_z = z \cdot a = w \neq 0.$$

Otherwise, in view of Lemma 1.5, f_k can be rewritten in the form

$$f_k = (\alpha_k (x_1 x_2) L_{x_3} + \beta_k (x_2 x_3) L_{x_1} + \gamma_k (x_3 x_1) L_{x_2}) R (LR)^{n-1},$$

where $\alpha_k = \beta_k + \alpha_{2,k}$ and $\gamma_k = \beta_k - \alpha_{3,k}$. If $\alpha_k, \beta_k, \gamma_k$ are not simultaneously zero, then by virtue of the restriction $\text{char}(\mathcal{F}) \neq 2$, one of their pairwise sum is not zero as well. Suppose, for example, that $\alpha_k + \beta_k \neq 0$; then we have

$$\frac{1}{\alpha_k + \beta_k} \tilde{f}_{2,k} = (\alpha_k y \cdot v + \beta_k v \cdot y) (L_y R_y)^n L_z = a (L_y R_y)^n L_z = z \cdot a = w \neq 0.$$

Thus all the scalars in the considered expression of f modulo Φ_d prove to be zeros. \square

Lemma 6.3. *If $f = 0$ in $G(\mathcal{A}')$ for some $f \in \Phi_d$, then $f \in \Phi'_d$.*

Proof. By Lemmas 5.1 and 5.4, we can write down f in the form

$$f \equiv \sum_{j=3}^{d-1} \sum_{i=2}^{j-1} \alpha_{i,j} \phi_n(1, i, j, d) + \beta_j \sum_{j=3}^{d-1} \phi_n(1, 2, d, j) \pmod{\Phi'_d},$$

where $\alpha_{i,j}, \beta_j \in \mathcal{F}$. By $\tilde{f}_{i,j}$ denote the value of \tilde{f} on the following elements of \mathcal{A}' :

$$x_i = x_j = z, \quad x_k = y \quad \text{for } k \in [1, d] \setminus \{i, j\}.$$

Suppose $\alpha_{i,j} \neq 0$; then $\tilde{f}_{i,j}$ is proportional to the element

$$z^2(L_y R_y)^{n-1} L_y = a(L_y R_y)^{n-1} L_y = y \cdot a = v \neq 0.$$

Otherwise, f gets the form

$$f \equiv \sum_{j=3}^{d-1} \beta_j \phi_n(1, 2, d, j) \pmod{\Phi'_d}.$$

In the case $\beta_j \neq 0$, by the similar calculations, we obtain $\tilde{f}_{d,j} \neq 0$.

Therefore, $f \in \Phi'_d$. □

Lemma 6.4. *If $f = 0$ in $G(\mathcal{A}')$ for some $f \in \Phi'_d$, then $f = 0$ in \mathfrak{M} .*

Proof. By Lemma 5.5, we can write down f in the form

$$f = \sum_{k=4}^{d-1} \sum_{j=3}^{k-1} \sum_{i=2}^{j-1} \alpha_{i,j,k} \varphi_n(1, i, j, d, k) + \sum_{k=4}^{d-1} \sum_{j=3}^{k-1} \beta_{j,k} \varphi_n(1, 2, d, j, k),$$

where $\alpha_{i,j,k}, \beta_{j,k} \in \mathcal{F}$. By $\tilde{f}_{i,j,k}$ denote the value of \tilde{f} on the following elements of \mathcal{A}' :

$$x_i = x_j = x_k = z, \quad x_\ell = y \quad \text{for } \ell \in [1, d] \setminus \{i, j, k\}.$$

Assume that $\alpha_{i,j,k} \neq 0$; then $\tilde{f}_{i,j,k}$ is proportional to the element

$$z^2(L_y R_y)^{n-1} L_z = z \cdot a = w \neq 0.$$

Otherwise, f gets the form

$$f = \sum_{k=4}^{d-1} \sum_{j=3}^{k-1} \beta_{j,k} \varphi_n(1, 2, d, j, k).$$

In the case $\beta_{j,k} \neq 0$, by the similar arguments, we obtain $\tilde{f}_{d,j,k} \neq 0$.

Therefore all the scalars in f are zeros. □

Lemmas 6.1–6.4 yield $\tilde{\mathfrak{M}} = \text{Var } G(\mathcal{A}') = \text{Var } \tilde{G}_2(\mathcal{V})$. Theorem is proved.

Remark. The variety \mathfrak{M} doesn't satisfy the condition of minimality, i. e. there are proper subvarieties of \mathfrak{M} that are also non-finitely based. For instance, it follows from the proof of the Theorem that by setting all the double ϕ -words equal to zero we distinguish the proper non-finitely based subvariety of \mathfrak{M} .

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